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On the finite space with a finite group action II

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1 Introduction

The purpose of our presentation was to apply the finite topology theory to the subgroup complex theory. A *finite T_0 -space* is a topological space having finitely many points with the T_0 -separation axioms, that is, for each pair of distinct two points, there exists an open set containing one but not the other. Many well-known properties about finite T_0 -spaces may be found in [1], [2] and [5]. Moreover we consider the finite space with a finite group G -action, called a *finite T_0 - G -space*.

On the other hand, we are interested in homotopy properties on subgroup complexes of a finite group. Let G be a finite group and p a prime factor of the order of G . Let $O_p(G)$ be the maximal normal p -subgroup of G . The Bouc poset (= partially ordered set) $B_p(G)$ of a finite group G is the subposet of $S_p(G)$ with $O_p(N_G(P)) = P$, where $N_G(P)$ is the normalizer of P and $S_p(G)$ is the poset of the non-trivial p -subgroups of G ordered by inclusion. We remark that the Bouc poset $B_p(G)$ contains all Sylow p -subgroups of G . Let $\Delta(B_p(G))$ denote the order complex of $B_p(G)$, that is, the vertices are the elements of $B_p(G)$ and the n -simplices are the chains of p -subgroups of $B_p(G)$ of length n . This simplicial complex is called the *Bouc complex* of G at p .

Quillen examined the simplicial complex $\Delta(S_p(G))$ associated with the poset $S_p(G)$. In particular, let us take a finite solvable group G . The main theorem of his paper [4] is that $\Delta(S_p(G))$ is contractible if and only if there is a non-trivial normal p -subgroup. Our study is motivated by this result.

McCord's result [3, Theorem 2] provides deep insight into understanding relations between finite T_0 -spaces and finite simplicial complexes. For a finite T_0 -space X , we can define the order complex $\Delta(X)$. Let $|\Delta(X)|$ be the geometric realization of $\Delta(X)$.

Proposition 1.1. *There exists a weak homotopy equivalence $\mu_X : |\Delta(X)| \rightarrow X$. Moreover, each map $\varphi : X \rightarrow Y$ between finite T_0 -spaces defines a simplicial map $\Delta(\varphi) : \Delta(X) \rightarrow \Delta(Y)$ by $\Delta(\varphi)(x) = \varphi(x)$, and $\varphi \circ \mu_X = \mu_Y \circ |\Delta(\varphi)|$ where $|\Delta(\varphi)| : |\Delta(X)| \rightarrow |\Delta(Y)|$ is a continuous map induced by $\Delta(\varphi)$.*

Corollary 1.2. *Let $\varphi : X \rightarrow Y$ be a map between finite T_0 -spaces. Then φ is a weak homotopy equivalence if and only if $|\Delta(\varphi)| : |\Delta(X)| \rightarrow |\Delta(Y)|$ is a homotopy equivalence.*

Then we show the following:

Theorem A. *Let G be a finite nilpotent group and p any prime factor of the order of G . Then $\Delta(B_p(G))$ is contractible.*

We apply McCord's theorem to give a very short, purely topological proof of the above result.

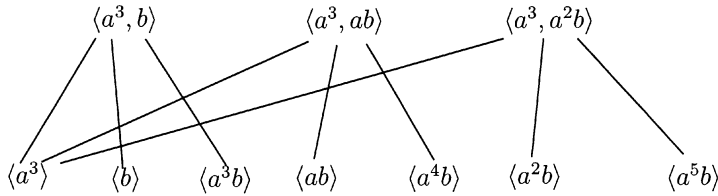
2 Some examples of Bouc posets

For the convenience of the reader, we present some examples of Bouc posets.

Example 2.1. Take $G = D_{12}$, the dihedral group of order 12, and $p = 2$. We can give the abstract presentation of G by the generators and relations:

$$G = \langle a, b \mid a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle;$$

where these represent a rotation and a reflection, when G is regarded concretely as the group of a regular hexagon. We find three Sylow 2-subgroups of order 4: $\langle a^3, b \rangle$, $\langle a^3, ab \rangle$, $\langle a^3, a^2b \rangle$, and the minimal members are generated by 7 involutions: $\langle a^3 \rangle$, $\langle b \rangle$, $\langle ab \rangle$, $\langle a^2b \rangle$, $\langle a^3b \rangle$, $\langle a^4b \rangle$, $\langle a^5b \rangle$. Thus the poset diagram for $S_2(D_{12})$ is given by:

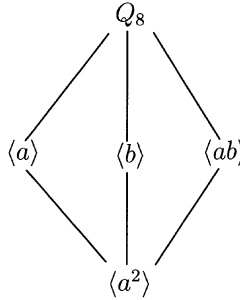


Observe that each of three Sylow 2-subgroups is not the normal subgroup of G and the center $Z(G)$ of G equals $\langle a^3 \rangle$. Therefore $B_2(G) = \{\langle a^3, b \rangle, \langle a^3, ab \rangle, \langle a^3, a^2b \rangle, \langle a^3 \rangle\}$.

Example 2.2. Take $G = Q_8$, the quaternion group of order 8, and $p = 2$. We can give the abstract presentation of G by the generators and relations:

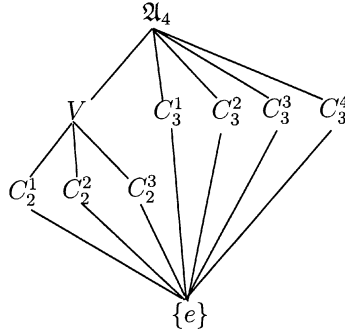
$$G = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle.$$

We find three Sylow 2-subgroups of order 4: $\langle a \rangle$, $\langle b \rangle$, $\langle ab \rangle$, and each of these three Sylow 2-subgroups contains the unique cyclic subgroup $\langle a^2 \rangle$. Thus the poset diagram for $S_2(Q_8)$ is given by:



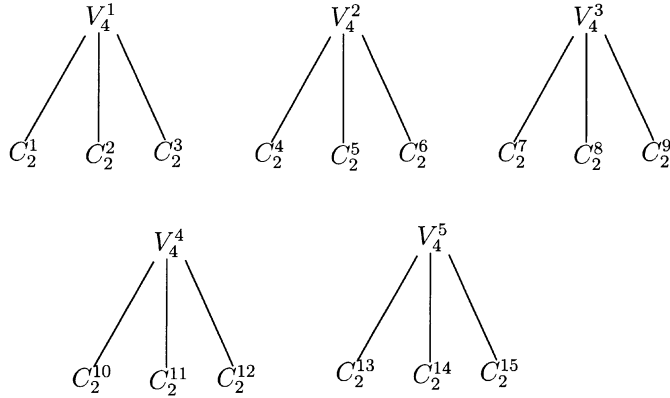
Since any subgroup of G is normal, so that $B_2(G) = \{Q_8\}$.

Example 2.3. Take $G = \mathfrak{A}_4$, the alternative group of letter 4. We find one Sylow 2-subgroup of order 4 and four Sylow 3-subgroups of order 3. The subgroups diagram for \mathfrak{A}_4 is given by:



Here V is a Klein group, each C_3^i ($i = 1, 2, 3, 4$) is a distinct cyclic group of order 3, and each C_2^j ($j = 1, 2, 3$) is a distinct cyclic group of order 2. Then $B_2(G) = \{V\}$ and $B_3(G) = \{C_3^1, C_3^2, C_3^3, C_3^4\}$.

Example 2.4. Take $G = \mathfrak{A}_5$, the alternative group of letter 5, and $p = 2$. By easy observation, we find five Sylow 2-subgroups of order 4, and each Sylow 2-subgroup contains three cyclic groups of order 2. Thus the poset diagram for $S_2(\mathfrak{A}_5)$ is given by:



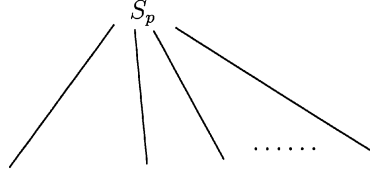
Here each V_4^i ($1 \leq i \leq 5$) is a distinct Klein group, each C_2^j ($1 \leq j \leq 15$) is a distinct cyclic group of order 2. Then $B_2(\mathfrak{A}_5) = \{V_4^1, V_4^2, V_4^3, V_4^4, V_4^5\}$.

3 Proof of Theorem A

We address to the article written by Barmak and cited in Bibliography. Stong studied equivariant homotopy theory for finite T_0 -spaces [6]. Let G be a finite group. A finite T_0 -space with a G -action is called a *finite T_0 - G -space*. Any finite T_0 - G -space X has a core which is G -invariant and an equivariant strong deformation retract of X . Such a core is

called a G -core. See our general reference Barmak [1, p106] for details. Note that a finite T_0 - G -space is contractible if and only if its G -core consists of a point. We remark that $B_p(G)$ is a finite T_0 - G -space by conjugation.

Proof of Theorem A If G is a finite nilpotent group, then G has a unique Sylow p -subgroup S_p . The poset diagram for $B_p(G)$ is given by:



By this diagram, the G -core of $B_p(G)$ is $\{S_p\}$, and so $B_p(G)$ is contractible. By McCord's theorem (Proposition 1.1), there exists the following commutative diagram:

$$\begin{array}{ccc}
 |\Delta(B_p(G))| & \xrightarrow{|\Delta(f)|} & |\Delta(\{S_p\})| \\
 \mu_{B_p(G)} \downarrow & & \downarrow \mu_{\{S_p\}} \\
 B_p(G) & \xrightarrow{f} & \{S_p\}
 \end{array}$$

where $f : B_p(G) \rightarrow \{S_p\}$ is homotopy equivalent. By Corollary 1.2, map $|\Delta(f)| : |\Delta(B_p(G))| \rightarrow |\Delta(\{S_p\})|$ is also homotopy equivalent. Therefore $|\Delta(B_p(G))|$ is contractible, that is, $\Delta(B_p(G))$ is contractible. \square

Corollary B. *Let pq be the order of G such that p and q are distinct primes with $p > q$. Then $\Delta(B_p(G))$ is contractible.*

Proof. The number of Sylow p -subgroups of G is equivalent to 1 modulo p . Moreover it is also the divisor of pq . Therefore the number of Sylow p -subgroups of G is 1, and so the Sylow p -subgroup is normal. \square

For example, take $G = \mathfrak{S}_3$, the symmetric group of letter 3. Then $\Delta(B_3(\mathfrak{S}_3))$ is contractible.

4 Concluding remarks

Lemma 4.1. *A contractible finite T_0 - G -space has a point which is fixed by the action of G .*

Proof. A contractible finite T_0 - G -space has a G -core, i.e. a point, which is G -invariant. \square

We showed the followig result in [2].

Lemma 4.2. *Let X be a finite T_0 - G -space. Then $|\Delta(X)|/G$ is homotopy equivalent to $|\Delta(X)/G|$.*

Suppose that $B_p(G)$ is contractible. Then lemma 4.1 claims that G has a normal p -subgroup. Moreover the orbit space $B_p(G)/G$ of $B_p(G)$ is a finite T_0 -space and also contractible.

Proposition 4.3. *Let $|\Delta(B_p(G))/G|$ be the geometric realization of $\Delta(B_p(G))/G$. If $B_p(G)$ is contractible, $|\Delta(B_p(G))/G|$ is also contractible.*

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